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ON n-EXTENDABLE GRAPHS

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A graph G is said to be n-*extendable* if it is connected, has a set of n independent lines and every set of n independent lines in G extends to (i.e. is a subset of) a perfect matching of G. Nearly all n-extendable graphs ($n \geq 2$) are shown to be $(n-1)$-extendable and $(n+1)$-connected. The special cases of 2- and 3-extendable graphs and their relationships with bicritical and elementary bipartite graphs are then studied.

1. Introduction

A *perfect matching* (hereafter "p.m."), or 1-*factor*, of a graph G is a set of independent lines which together cover all the points of G. One of the most important recent papers dealing with the structure of graphs which have p.m.'s is due to Lovász [6]. A problem which has attracted much attention in graph theory—and one which is not yet completely settled—is the following: given a graph G with a p.m., to determine a greatest lower bound on the number of *different* p.m.'s G must contain. Motivated by his studies of this problem, Lovász began to develop a structure theory for graphs with p.m.'s. Call a line of graph G *allowed* if it lies in some p.m. of G. A graph G is *elementary* if its allowed lines form a connected subgraph of it. A graph is *bicritical* if the deletion of any two points of G results in a graph with a p.m. Lovász showed that in a certain sense any graph with a p.m. could be constructed using only elementary bipartite graphs and bicritical graphs as building blocks.

Subsequently, the study of these two classes of graphs has been continued by Lovász and the present author [8, 9, 10]. See also the survey article by Lovász [7]. In the present paper, a study of n-extendable graphs is undertaken. A graph G is n-*extendable* if it is connected, contains a set of n independent lines and every set of n independent lines extends to (i.e. is a subset of) a p.m. of G. One readily sees that a graph is 1-extendable iff it is elementary without forbidden lines. C.H.C. Little proved a nice theorem about these [5]. In particular, he shows that in such a graph any two lines lie on a cycle which alternates with respect to some p.m.

In Section 2 of this paper we present some results of a preliminary nature. First we give some examples of n-extendable graphs and prove the existence of a large family of such graphs. The proof of the important fact that n-extendability implies $(n-1)$-extendability in nearly all cases concludes the section.

In Section 3 we obtain the result that for all $n \leqslant \frac{1}{2}p-1$, an n-extendable graph is necessarily $(n+1)$-connected.

In Section 4 we classify 2-extendable graphs in the sense that we prove such graphs must be either elementary bipartite or bicritical. (That no graph is both bipartite and bicritical is immediate.) We conclude with some counterexamples to further plausible conjectures.

For any terminology used, but not defined, in this paper, the reader is referred to Lovász [6] and Harary [4]. All graphs treated in this paper are assumed to be connected unless otherwise specified.

2. Preliminary results

In general, the family of n-extendable graphs is quite a large one. For example, when $n=2$, the tetrahedron, the cube, the dodecahedron and the icosahedron (but not the octahedron) are 2-extendable as are all complete (equi-) bipartite graphs $K(r,r)$ where $r \geqslant 2$ in both cases. Our first theorem, qualitatively viewed, says roughly that if the minimum degree $\delta(G)$ of any graph G is large enough with respect to n and $|V(G)|$ (denoted by p hereafter unless confusion seems likely), then G is n-extendable.

Theorem 2.1. *Let n, k and p be positive integers with p even, $p \geqslant 4$ and $\frac{1}{2}p+n \leqslant k \leqslant p-1$. Then any graph G with p points and $\delta(G) \geqslant k$ is n-extendable. Moreover, the lower bound for k is best possible.*

Proof. Let n, k and p be as in the hypothesis and let G be a graph on p points with $\delta(G) \geqslant k$. Then $\delta(G) \geqslant k \geqslant \frac{1}{2}p+n > \frac{1}{2}p$ and so by Dirac's theorem on Hamiltonian cycles [1] G has a Hamiltonian cycle and since p is even, G must then have a p.m. containing $\frac{1}{2}p \geqslant n+1 > n$ independent lines.

Now let X be any set of n independent lines in G and let $G' = G - V(X)$. Then G' has $p-2n$ points and $\delta(G) \geqslant k \geqslant \frac{1}{2}p+n$, so $\delta(G') \geqslant \delta(G) - 2n \geqslant \frac{1}{2}p - n = \frac{1}{2}|V(G')|$ and again by the Dirac theorem, G' has a Hamiltonian cycle. But since p is even, $p-2n = |V(G')|$ is even and G' therefore must have a p.m. Thus G has a p.m. containing X.

To see that the bound is sharp, let $p = 2r$ be any even integer, with $r \geqslant 2$. Form a graph $G(p,n)$ as follows: join every point of the complete graph K_{r+n-1} to each of $r-n+1$ independent points. Now let $L = \{a_1b_1, \ldots, a_nb_n\}$ be any set of n independent lines in the K_{r+n-1} part of $G(p,n)$. Then in the graph $G'(p,n) = G(p,n) - \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ there remain $r+n-1-2n = r-n-1$ points which cannot be matched onto the remaining $r-n+1$ independent points. Thus $G(p,n)$ is not n-extendable. Clearly $\delta(G(p,n)) = r+n-1 - \frac{1}{2}p+n-1 = \deg u$, where u is any of the $r-n+1$ independent points in $G(p,n)$. $\square$

On the other hand, connectivity large with respect to n does not in general

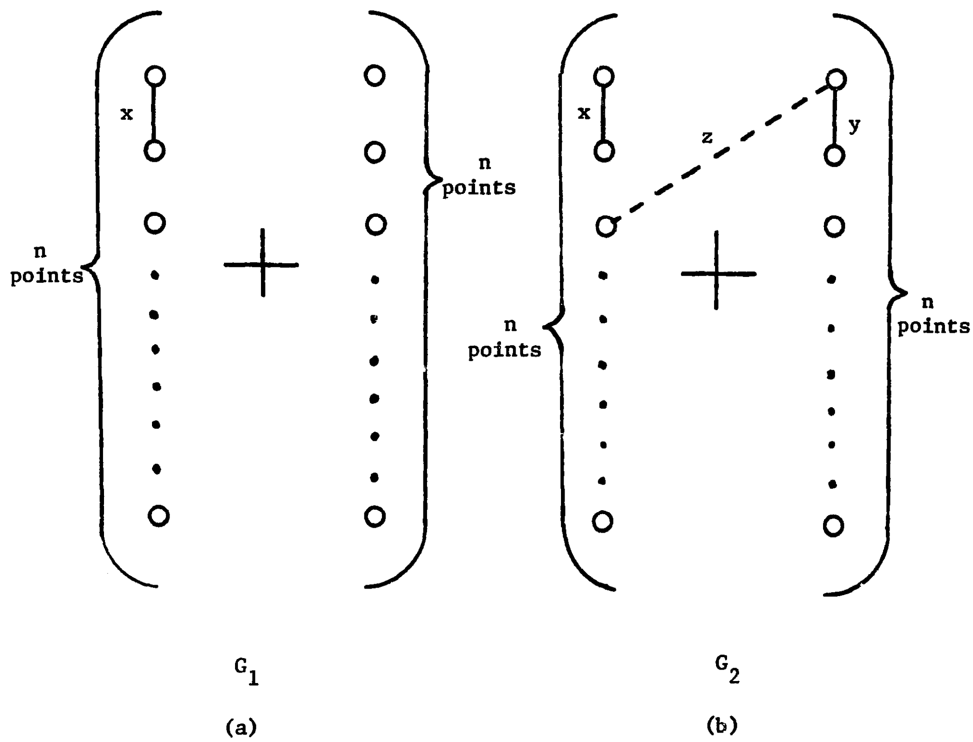


Fig. 1.

imply n-extendability. The graph G_1 consisting of $K(n, n)$ with a single line x added (cf. Fig. 1(a)) is n-connected, has a p.m., but is not 1-extendable (x does not extend). Similarly, graph G_2 consisting of $K(n, n)$ with two lines x and y added as in Fig. 1(b) is n-connected, is 1-extendable, but not 2-extendable (x and z do not extend.). (In both cases the large + sign means all points on the left are joined to all points on the right.)

In fact, even if G is bipartite, large connectivity does not imply n-extendability, even if n is small. We illustrate with two more examples.

First, construct a bipartite graph G_3 with bipartition (A, B) as follows. Let $A = \{a\} \cup A_1 \cup A_2$ where $|A_1| = n$ and $|A_2| = n-1$ and let $B = \{b\} \cup B_1 \cup B_2$ with $|B_1| = n-1$ and $|B_2| = n$. Then $|A| = |B| = 2n$. The adjacencies are as follows: each point of $A_2 \cup \{a\}$ is adjacent to each point of B and thus has degree $= 2n$, each point of A_1 is adjacent to $B_1 \cup \{b\}$ and hence has degree $= n$. It follows that each point of $B_1 \cup \{b\}$ has degree $2n$ and each point of B_2 has degree n. Clearly G_3 has a p.m. obtained by matching A_1 onto $\{b\} \cup B_1$ and each point of $\{a\} \cup A_2$ onto B_2. On the other hand, $x = ab$ lies in no p.m. for if it does, $G_3' = G_3 - a - b$ has a p.m. But $|\Gamma_{G_3'}(A_1)| = |B_1| = n-1 = |A_1| - 1 < |A_1|$, where $\Gamma_{G_3'}(A_1)$ denotes the set of points in G_3 adjacent to some point in A_1. Thus the classical theorem of P. Hall on matchings in bipartite graphs [3] is contradicted. So G_3 is an n-connected bipartite graph which is not 1-extendable.

Secondly, a slight modification of G_3 yields another graph G_4 which is n-connected ($n \geq 2$), bipartite and 1-extendable, but not 2-extendable. Let the bipartition (A, B) of G_4 be as follows: $A=\{a, c\} \cup A_1 \cup A_2$ with $|A_1|=n$, $|A_2|=n-2$, $B=\{b, d\} \cup B_1 \cup B_2$ with $|B_1|=|B_2|=n-1$. Thus $|A|=|B|=2n$. The adjacencies are as follows: each point of $A_2 \cup\{a, c\}$ is adjacent to B and is thus of degree $2n$, while each point of A_1 is adjacent to $B_1 \cup\{b, d\}$ and therefore has degree $n+1$. It follows that each point of $B_1 \cup\{b, d\}$ has degree $2n$ and each point of B_2 has degree n.

Using Hall's theorem again, $G_4'=G_4-a-b-c-d$ has no p.m. since $|\Gamma_{G_4'}(A_1)|=n-1=|A_1|-1<|A_1|$. It is routine to check that G_4 is extendable.

It is a trifle vexing to note that there are graphs which are n-extendable, but not $(n-1)$-extendable. For $n \geqslant 2$, a trivial example is the path of length $2n-1$. Here of course, $p=2n$. We now proceed to show, however, that if $p \geqslant 2n+2$, then n-extendability does imply $(n-1)$-extendability.

Theorem 2.2. *Let positive integers n and p be given where $n \geqslant 2$, p is even and $p \geqslant 2n+2$. Then if G is an n-extendable graph on p points, G is also $(n-1)$-extendable.*

Proof. Suppose n, p and G satisfy the hypotheses of the theorem, but suppose G is not $(n-1)$-extendable. In particular, let X be a set of $n-1$ independent lines which do not extend to a p.m. and let M be any p.m. of G. Then $M \oplus X$ (where $\oplus$ denotes symmetric difference) consists of some number of even cycles together with at least two alternating paths, each of which has both its first and last lines in M. Let P be the line set of one such path. Then $P \oplus X$ is a set of n independent lines which can be extended to a p.m.. Moreover, this p.m. will contain at least one line e not in $P \oplus X$, since $|P \oplus X|=n$ and $p \geqslant 2n+2$. But then $X \cup\{e\}$ is a set of n independent lines which extends to a p.m. containing X, a contradiction. □

3. Connectivity of n-extendable graphs

In the introduction we saw that if the connectivity of a graph is at least $\frac{1}{2}p+n$ then it must be n-extendable. However, many graphs with much smaller connectivity are n-extendable as well. Hence it seems appropriate to ask if an n-extendable graph has a non-trivial lower bound (involving n) on its connectivity. We proceed to prove the answer affirmative.

Lemma 3.1. *Every 1-extendable graph G $(G \neq K_2)$ is 2-connected.*

Proof. Let G be 1-extendable, and suppose v is a cutpoint and let $C_1, \ldots, C_k$ $(k \geqslant 2)$ be the components of $G-v$. Let x denote any line joining v to a point of C_1. Then x extends to a p.m. of G implying $|V(C_1)|$ is odd (and $|V(C_2)|, \ldots, |V(C_k)|$ are all even). But then if y is any line joining v to C_2, it follows similarly that $|V(C_2)|$ is odd, a contradiction. □

We remark in passing, if the reader has not already noticed, that n-extendable graphs on $2n$ points have not really been treated thus far. They are, by definition, precisely those graphs which contain a p.m. and these graphs are precisely those characterized by the well-known theorem of Tutte [11]. The reader should also note that these are not necessarily $(n-1)$-extendable. Earlier we gave a trivial family of examples of this kind of behaviour, namely the paths of length $2n-1$. For $n=2$ there are precisely five 4-point graphs which are 2-extendable and among these precisely two are 1-extendable, while three are not. The reader may easily discern these.

The reader may readily show also that an n-extendable graph G on $p=2n$ points is $(n-1)$-extendable if and only if for every pair u, v of non-adjacent points, $G-u-v$ has no p.m.

The existence of these few "troublesome" graphs indicates the reason for the necessity of the requirement that $p \geqslant 2n+2$ in the hypothesis of Theorem 2.3 and also in the theorem now to follow.

Theorem 3.2. *Let n be a positive integer. If G is an n-extendable graph on $p \geqslant 2n+2$ points, G is $(n+1)$-connected.*

Proof. The proof is by induction on n. For the case $n=1$, the result follows from Lemma 3.1.

Now suppose the result is true for some $n-1 \geqslant 0$. We proceed to show the result valid for n. Hence let G be an n-extendable graph with $p \geqslant 2n+2$. Then by Theorem 2.2, G is also $(n-1)$-extendable and thus by the induction hypothesis G is n-connected.

Suppose now that G is not $(n+1)$-connected. Thus G contains a cutset of points, S, of cardinality n. Let $C_1, \ldots, C_k$ $(k \geqslant 2)$ be the components of $G-S$. Note first that $|V(C_1)|+\cdots+|V(C_k)|=|V(G)|-|S| \geqslant 2n+2-n=n+2>n+1$ so by a variation of Menger's theorem due to Dirac [2] there are n point-disjoint paths in G each having one endpoint in S and the other in $\bigcup_{i=1}^{k} V(C_i)$. It then follows that there is a set L of n independent lines joining S to $\bigcup_{i=1}^{k} V(C_i)$.

First, suppose some C_i has $|V(C_i)| \geqslant n$. Then in fact, Dirac's theorem says that there are n disjoint paths in G joining S and $V(C_i)$ and hence there is a set L of n independent lines joining C_i and S which, of course, must cover S. Now let x_1 be any line of L, $x_1=c_1s_1$; $s_1 \in S$. Since $S-\{s_1\}$ is not a cutset of G, there is a line x_1' joining s_1 to a point of C_j for some $j \neq i$. Moreover, if $L'=L-\{x_1\} \cup \{x_1'\}$, L' is also a set of n independent lines in G.

Now suppose n is even. Then since there is a p.m. of G containing L, $|V(C_i)|$ is even. On the other hand, since L' also extends to a p.m. of G, it follows that $|V(C_i)|$ is odd, a contradiction. A similar contradiction is reached upon assuming n to be odd.

So we may suppose that each of $|V(C_1)|, \ldots, |V(C_k)| \leqslant n-1$. Now assume that for some $i, 2 \leqslant |V(C_i)| = m \leqslant n-1$ and let $V(C_i) = \{u_1, \ldots, u_m\}$. Let $R_1 = \{u_1, \ldots, u_{m-1}\}$. We have $|V(G) - S - V(C_i)| = |V(G)| - |S| - |V(C_i)| \geqslant 2n + 2 - n - m = n - m + 2$, so choose any set $R_2 \subset V(G) - S - V(C_i)$ such that $|R_2| = n - m + 1$. Then $|R_1 \cup R_2| = m - 1 + n - m + 1 = n$ and again using Dirac's theorem there are n disjoint paths in G joining S to $R_1 \cup R_2$. But then there is a set L of n lines joining some $m-1$ points of C_i and some $n-(m-1)$ points of $V(G) - S - V(C_i)$ to S. Let u denote the single point of C_i not covered by L. Now L covers S and extends to a p.m., M, of G. But then M cannot cover u, a contradiction.

Thus for all i, $1 \leqslant i \leqslant k$, $|V(C_i)| = 1$. But since G has a p.m., it follows that $k \leqslant n$. Hence $|V(G)| \leqslant 2n$ contradicting the earlier assumption that $p \geqslant 2n+2$. □

For some values of p and n it is easy to see that the bound in Theorem 3.2 is sharp. For example, let n be any positive integer $\geqslant 2$ and let $p = 2n+2$. Then the bipartite graphs $K(n+1, n+1)$ are n-extendable, have $q = (n+1)^2$ lines, are $(n+1)$-connected, but not $(n+2)$-connected. Certain other extremal families for certain specific values of n and p have been found by the author. However for arbitrary positive integral n and p, where p is unrelated to n other than by the inequality $p \geqslant 2n+2$, the problem remains unsettled.

Given a graph G and integers p and n as in the hypotheses of the preceding theorem, it follows easily that those points (if any) of degree $n+1$ (the minimum possible) must have independent neighborhoods.

Corollary 3.3. *If n and p are positive integers, $p \geqslant 2n+2$, if G is an n-extendable graph on p points and if u is a point of degree $n+1$ in G, then $\Gamma_G(u)$ is an independent set.*

Proof. Let u be a point of degree $n+1$ and let $\Gamma_G(u) = \{v_1, \ldots, v_n, v_{n+1}\}$. Since $p \geqslant 2n+2$ choose any n points, say $W = \{w_1, \ldots, w_n\}$, in $V(G) - \Gamma_G(u) - \{u\}$. Since G is $(n+1)$-connected, by Dirac's theorem we have $n+1$ point-disjoint paths joining $\Gamma_G(u)$ and $W \cup \{u\}$. Hence there are $n+1$ independent lines $y_1 = v_1 u$, $y_2 = v_2 w'_1, \ldots, y_{n+1} = v_{n+1} w'_n$.

Now suppose $\Gamma_G(u)$ is not independent; i.e., suppose without loss of generality that $v_1 v_2 \in E(G)$. Then $\{v_1 v_2, y_3, \ldots, y_{n+1}\}$ is a set of n independent lines which cannot extend to a p.m. of G covering u, a contradiction. □

4. Some classification results

In this section we shall study how the concept of n-extendability fits in with the properties of being elementary and being bicritical (concepts defined in the introduction).

The following characterization of bicritical graphs due to Lovász [6] will be helpful to us.

Theorem 4.1. [6, Proposition 3.5] *Let G be a graph on an even number of points. Then G is bicritical iff for every set $S \subset V(G)$ with $|S| \geqslant 2$, $G-S$ has at most $|S|-2$ odd components.*

We can now classify 2-extendable graphs via the following theorem. It is important to realize at the outset that no bicritical graph can be bipartite so that the two properties mentioned in this theorem, namely the properties of being bicritical and being elementary bipartite, are mutually exclusive.

Theorem 4.2. *Let G be 2-extendable with $p \geqslant 6$ points. Then G is either bicritical or elementary bipartite.*

Proof. G is elementary by Theorem 2.2. Suppose G is not bicritical. Hence by Theorem 4.1, there exists a set $S \subseteq V(G)$ with $|S| \leqslant c_0(G-S)+1$ and hence by parity, $|S| \leqslant c_0(G-S)$. Since G contains a p.m., by Tutte's well-known theorem [11], we must have $|S| = c_0(G-S)$. (Here $c_0(G-S)$ denotes the number of odd components in $G-S$). It is then immediate that every p.m. of G must therefore match each point of S with a point of a different odd component of $G-S$. But then since G has no forbidden lines, S is an independent set in G and $G-S$ has no even components.

Finally, we need only show that each component of $G-S$ consists of a single point. Suppose, to the contrary, that N is an odd component with at least three points. Again, since G is 2-connected, there must be two independent lines x' and y' joining N to S. Since G is 2-extendable, $\{x', y'\}$ extends to a p.m. M' of G. (Actually, by parity, M' must contain at least three lines joining N to S.) But each of the other $|S|-1$ odd components must have a line of M' joining it to a point of S. Hence M' contains at least $3+c_0(G-S)-1 = 3+|S|-1 = |S|+2$ lines incident with S, a contradiction.

Thus G is bipartite with bipartition (S, T) where $|S| = |T|$. $\square$

We point out that a 2-extendable bicritical graph need not be minimal bicritical, for graph G_5 of Fig. 2(a) is such a graph. To see that G is not minimal, observe that graph H in Fig. 2(b) is a spanning bicritical proper subgraph of G.

On the other hand, note that a 2-extendable graph which is elementary bipartite cannot, under any circumstances, be minimal elementary bipartite, unless it is the cycle on four points. This follows immediately from Theorem 5 of [9] which implies that a minimal elementary bipartite graph must have points of degree two. On the other hand, since for $p=4$ there are no minimal elementary bipartite graphs other than C_4 and since for $p \geqslant 6$ all 2-extendable graphs have minimum degree at least three, the conclusion follows.

Theorem 4.2 is best possible in the sense that there are 1-extendable graphs with $p \geqslant 6$ which are neither bicritical nor bipartite. The graph G_6 of Fig. 3 is such a graph. Note that G_6 is not bicritical since $G_6 - a - b$ has no p.m. (Of course, G_6 is not 2-extendable; e.g. $\{x, y\}$ does not extend.)

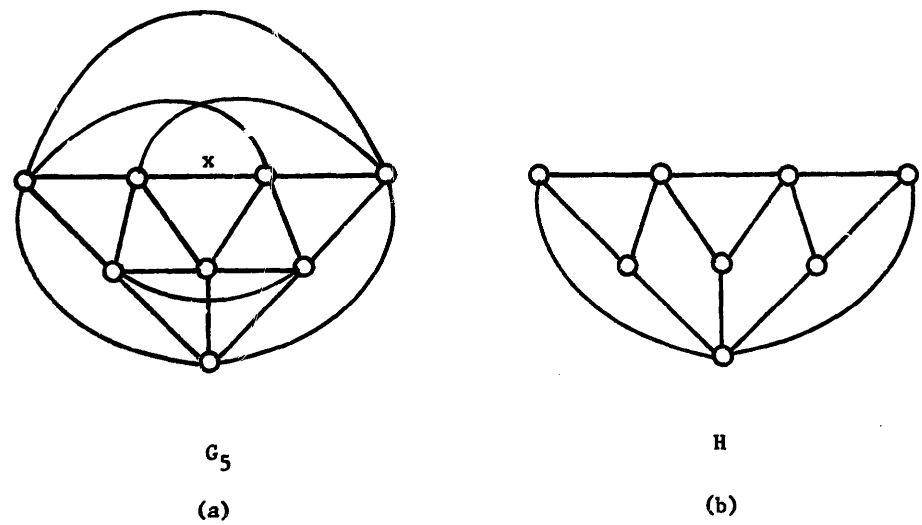


Fig. 2.

Let us note in passing two reasonable conjectures in the sense of providing some kind of converse result to Theorem 4.2. They are both false, however. In Fig. 4(a), G_7 is a 3-connected minimal bicritical graph which is not 2-extendable ($\{x, y\}$ does not extend). In Fig. 4(b), G_8 is a 3-connected elementary bipartite graph which is not 2-extendable ($\{x, y\}$ does not extend).

It is of interest, we think, to point out at this point that Corollary 11.1 of [9] says that any two lines of an elementary bipartite graph must lie on a *nice* cycle. (A subgraph H of an elementary graph G is *nice* if H is elementary and $V(G) - V(H)$ has a p.m.) But this does not mean that these two lines must extend to a p.m. of G. For example, in Fig. 4(b) lines x and y lie on a nice 14-gon, but they do not extend. The reader familiar with the "ear" terminology of [9] will note that G_2 is constructible by starting with the 14-gon containing x and y and then adding seven ears of length one (i.e. lines) joining points in opposite classes of the bipartition.

Now let us recall that a 2-extendable bicritical graph may or may not be minimal bicritical. For example, the graph G_5 of Fig. 2(a) is bicritical, but not

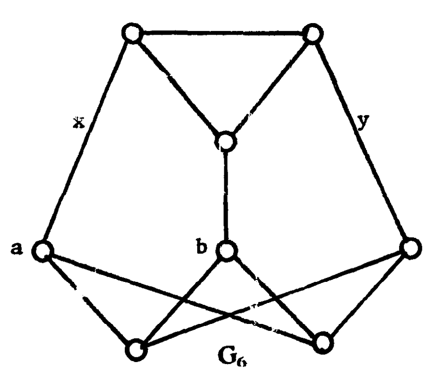
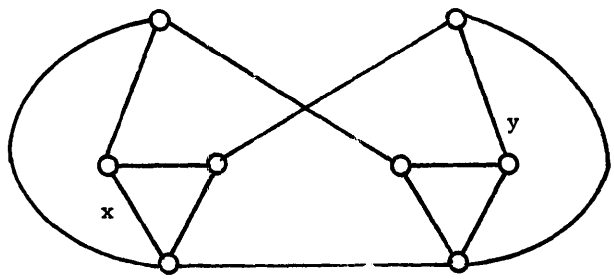
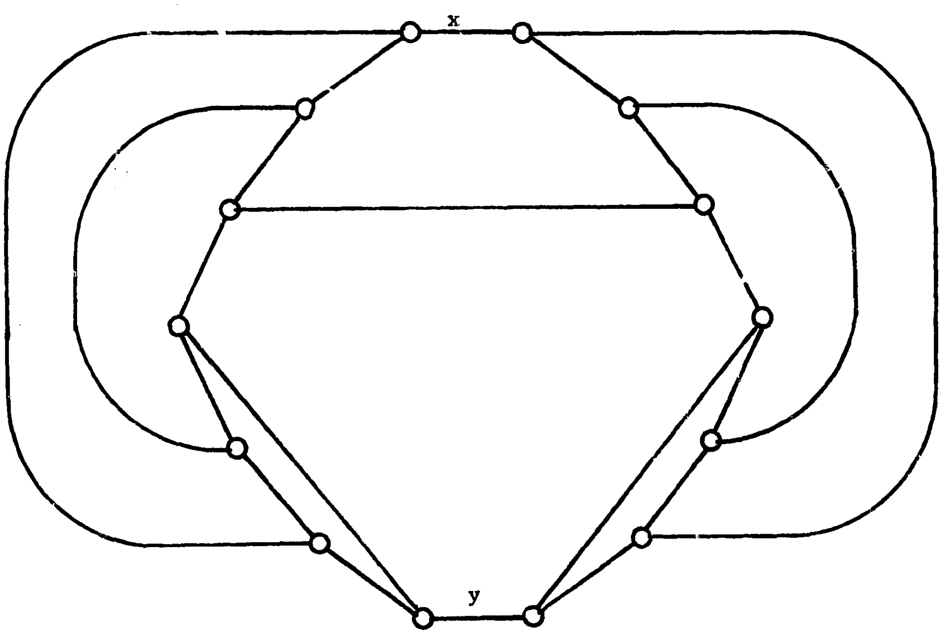


Fig. 3.



G_7

(a)



G_8

(b)

Fig. 4.

minimally so, whereas the dodecahedron is minimal bicritical. Each is, of course, 2-extendable. We conclude our discussion by showing that if a bicritical graph is 3-extendable, then it *cannot* be minimal. In fact we show more, namely that if any line is deleted from a 3-extendable bicritical graph (with $p \geqslant 8$) the resulting graph remains bicritical.

Theorem 4.3. *If G is 3-extendable and bicritical with $p \geqslant 8$ and if e is any line in G, then $G-e$ is again bicritical.*

Proof. Suppose G, p and e are as given in the hypothesis, but suppose $G-e$ is not bicritical. Then by Theorem 4.1 there is a set $S \subseteq V(G-e) = V(G)$, $|S| \geqslant 2$, such that $G-e-S$ has more than $|S|-2$ odd components. So $c_0(G-e-S)\geq |S|-1$ and again by parity, $c_0(G-e-S)\geq|S|$. But once again, by Tutte's theorem [11]. $c_0(G-e-S)=|S|$.

Since G is 4-connected by Theorem 3.2, it follows in fact that $|S|\geq 4$. Since G is bicritical, line e must join two points of different odd components C_1 and C_2 of $G-e-S$.

Since G is 4-connected there are at least four independent lines joining S to $C_1\cup C_2$. But then at least two of these four—say e_1 and e_2—meet $C_1\cup C_2$ in points different from the endpoints of e. Since G is extendable the lines e, e_1 and e_2 lie in some p.m. M of G. But since C_1 and C_2 are odd, $|V(C_1)|\geq 3$ and $|V(C_2)|\geq 3$ and hence M must contain at least three lines incident with each of C_1 and C_2. Among these lines at least two join C_1 to S and at least two others join C_2 to S.

On the other hand, M must match at least one point of each of the remaining odd components of $G-(C_1\cup C_2\cup S)$ to a distinct point in S. This contradicts the equation $|S|=c_0(G-e-S)$ and completes the proof. □

Of course there are many graphs which are both 3-extendable and bicritical; e.g. all K_{2n}'s for $n\geq 3$.

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